PULSATILE PIPE FLOW OF PSEUDOPLASTIC MATERIALS; THE FLOW ENHANCEMENT FOR RE $\ll$ 1

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Solution of the title problem for the power-law model of viscosity function is constructed by the method of small parameter in the region of small Reynolds numbers. The main result of the paper is a quantitative estimation of the values of Re, when the influence of inertia on flow enhancement may be quite neglected.

Substantial enhancement of the volumetric flow rate in a pipe can be achieved by superposition of shear oscillations on the basic steady flow under fixed steady pressure gradient. A quantitative measure of this effect, denoted often as "flow enhancement" (FE), is given by the ratio

$$I = Q_{osc}/Q_{std}\,, \tag{1}$$

where Q_{osc} stands for the volumetric flow rate under conditions of superposed oscillations and Q_{std} stands for the volumetric flow rate under steady conditions and the same mean pressure gradient. The extensive mathematical study of a number of constitutive models in the late ten years[1-3] has led to rather negative conclusion, that the primary constitutive cause of FE is exclusively a nonlinearity of viscosity function, while elastic memory effects may play only secondary role. On the other hand it was quantitatively shown[3,4], that inertia forces have a substantial influence on the extent of FE in the region of sufficiently high values of Reynolds number,

$$Re = (R/2)^{1+n}\, a^{1-n}\omega^{2-n}\varrho K^{-1}\,. \tag{2a}$$

Reynolds number is the only criterion, which demarcates the boundary-layer rheodynamic regime with strong inertia effects and the creeping rheodynamic regime, when the effect of inertia plays secondary role. This demarcation holds[3-5] in the full range of values of co-determinating rheodynamic criterion

$$Fr = a\omega^2/g\,. \tag{2b}$$

The oscillatory boundary-layer theory[3,4] was checked by experiments with kaoline suspensions[5], when FE has reached values $I = 10^{12}$, and the agreement has been fairly good.

In this paper we are dealing with the region of relatively small Re, when the inertia forces have rather secondary character. The main aim of the analysis is estimation of the critical values of Re for the power-law viscosity function, when the effect of inertia may be quite neglected. The possible influence of elastic memory effects is neglected for the above given reasons.

THEORETICAL

Let a non-Newtonian fluid with density ϱ and power-law viscosity function

$$\gamma = [\tau/K]^{m}, \quad m = 1/n \tag{3a,b}$$

periodically flows through the pipe with radius R, axially oscillating about its mean position with the instantaneous velocity of the pipe wall

$$v = a_0 \omega \cos(\omega t). \tag{4}$$

The fluid flows through the pipe due to gravity acceleration g_z and pressure gradient $-\partial_z p$, which pulsates about the mean value P_S with amplitude P_0 and frequency ω:

$$-\partial_z p = P_S - P_0 \sin(\omega t). \tag{5}$$

Introducing normalized quantities defined in the list of symbols the considered periodical flow situation can be described by the following boundary-value problem[4,5] for the field $W(Y, T)$ of the normalized oscillatory component of shear stresses:

$$M|1 + Fr\, W|^{m-1}\, \partial_T W = Y^{-m}\, \partial_Y Y^{-1}\, \partial_Y (Y^2 W) \tag{6}$$

with boundary conditions

$$Y W|_{Y=0} = 0, \quad \partial_Y (Y^2 W)|_{Y=1} = \sin T \tag{7a,b}$$

and with the evident condition of periodicity, $W(Y, T + 2\pi) = W(Y, T)$. Parameter M is defined by the relation

$$M = 4mRe^{m}Fr^{-m+1}. \tag{8}$$

By means of the field $W(Y, T)$ the wanted expression for FE according to the definition (*1*) may be found in the form[4,5]

$$I = (3 + m) \int_0^1 \mathbf{M}_T\{[1 + Fr\, W]^m\}\, Y^{2+m}\, dY\,, \tag{9}$$

where $\mathbf{M}_T\{\ldots\}$ stands for the time-averaging operator of a periodical function, $f(T + 2\pi) = f(T)$:

$$\bar{f} = \mathbf{M}_T\{f(T)\} = \frac{1}{2\pi} \int_{T_0}^{T_0 + 2\pi} f(T)\, dT\,, \tag{10}$$

where T_0 is arbitrary. Brackets in Eqs (*3a*), (*9*) and hereafter have been used for denoting an odd power-law function, $[x]^p = \mathrm{sign}\,(x)\, |x|^p$.

For further consideration it appears essential, that for $M = 0$, that is for $Re = 0$ at finite Fr and m, the boundary problem (*6*), (*7a,b*) has the extremely simple solution

$$W(Y, T) = G_0(Y)\, \phi_0(T) = \sin T\,, \quad Re = 0\,. \tag{11}$$

Regarding individual boundary problems as points in three-dimensional diagnostic space (Re, Fr, m), $Re \in \langle 0; \infty)$, $Fr \in \langle 0; \infty)$, $m \in \langle 1; \infty)$, the asymptote (*11*) represents the solution in the plane $Re = 0$ without the boundary lines $Fr = 0$ and $Fr = \infty$. Our aim is to find a perturbation continuation of the singular solution (*11*) into the thin layer $Re \ll 1$, close to the plane $Re = 0$.

This asymptotic approximation will be constructed by formal expansion into the functional series with respect to a small parameter M

$$W(Y, T) = \sum_{k=0}^{\infty} M^k G_k(Y; m)\, \phi_k(T; m, Fr)\,; \quad M \to 0\,, \tag{12}$$

where, according to Eq (*11*), $G_0 = 1$ and $\phi_0 = \sin(T)$ hold. The expansion will be constructed up to $k = 2$ including.

Substitution of the series (*12*) into the equation of motion (*6*) and expansion of the non-linear term on the left hand side of Eq (*6*) with respect to the powers of M leads directly to the identities

$$\phi_1(T) = (m\, Fr)^{-1}\, \partial_T [1 + Fr\, \phi_0]^m = \cos T\, |1 + Fr \sin T|^{m-1} \tag{13a}$$

$$\phi_2(T) = (m\, Fr)^{-2}\, \partial_T (|1 + Fr\, \phi_0|^{m-1} \phi_1) =$$
$$= \{-\sin T + 2(m-1)\, Fr \cos^2 T / (1 + Fr \sin T)\}\, |1 + Fr \sin T|^{2m-2}\,. \tag{13b}$$

To determine functions $G_1, G_2, \ldots$ we have to solve the consecutive set of second order differential equations

$$Y^{-m}\,\partial_Y Y^{-1}\,\partial_Y(Y^2 G_k) = G_{k-1}\,, \quad k = 1, 2, \ldots \tag{14}$$

with boundary conditions according to Eq (*7a,b*)

$$(YG_k)|_{Y=0} = 0\,, \quad \partial_Y(Y^2 G_k)|_{Y=1} = 0\,. \tag{15}$$

The solution is again simple:

$$G_1(Y) = -\frac{(m+3)/2 - Y^{m+1}}{(m+3)(m+1)}\,, \tag{16a}$$

$$G_2(Y) = \frac{(Y^{m+1} - (m+2))^2}{4(m+3)(m+2)(m+1)^2}\,. \tag{16b}$$

The obtained solution of second order with respect to M is interesting itself and offers a number of considerations on combined effect of inertia and nonlinear viscosity on the rise of strongly non-harmonic fluctuations of the shear stress. We will not regard these details and we will focuse our attention directly on the calculation of FE within framework of second order effects. Following this aim we will expand the nonlinear term $[1 + Fr\,W]^m$ with respect to the powers of the parameter M:

$$[1 + Fr\,W]^m = [1 + Fr \sin T]^m \left(1 + \frac{Fr(W - \sin T)}{1 + Fr \sin T}\right)^m \approx \tag{17a}$$

$$\approx [1 + Fr \sin T]^m \left(1 + \frac{MFrG_1\phi_1}{1 + Fr \sin T} + \frac{M^2 FrG_2\phi_2}{1 + Fr \sin T} + \ldots\right)^m \approx \tag{17b}$$

$$\approx [1 + Fr \sin T]^m + MmFrG_1 \cos T|1 + Fr \sin T|^{2m-2} -$$
$$- M^2 G_2 mFr \sin T|1 + Fr \sin T|^{3m-3} +$$
$$+ M^2(4G_2 + G_1^2)(m(m-1)/2)\,Fr^2 \cos^2 T[1 + Fr \sin T]^{3m-4}\,. \tag{17c}$$

To express I according to the definition (*9*) it is necessary in addition to integrate the expansion (*17c*) with relevant weight over the pipe cross-section and to average it according to the time. Performing these integrations we will make use of the following identities:

$$4(3 + m)\int_0^1 G_2 Y^{2+m}\,\mathrm{d}Y = (3 + m)\int_0^1 G_1^2 Y^{2+m}\,\mathrm{d}Y = p(m)/(4m^3)\,, \tag{18}$$

where

$$p(m) = \frac{4m^3(3m + 8)}{(m + 2)(3m + 5)(m + 3)}. \tag{19}$$

We will yet return to the calculation of time averages

$$\mu_0(m, Fr) = \mathbf{M}_T\{[1 + Fr \sin T]^m - 1\} \tag{20}$$

$$\mu_2(m, Fr) = \mathbf{M}_T\{Fr^{-1} \sin T|1 + Fr \sin T|^{3m-3} -$$

$$- \tfrac{5}{2}(m - 1) \cos^2 T[1 + Fr \sin T]^{3m-4}\}. \tag{21}$$

Making use of Eqs (*18*), (*20*), (*21*) and substituting the expansion (*17*) into the definition (*9*) we arrive to the asymptotic approximation of FE in the form

$$I = 1 + \mu_0(1 - \varepsilon_2 Re^{2m}) + O(Re^{4m}), \tag{22}$$

where

$$\varepsilon_2 = p(m) Fr^{-2m+4} \mu_2/\mu_0. \tag{23}$$

The term of the order $O(Re^m)$ in Eq. (22) vanishes, as the corresponding function ϕ_1, is similarly as $\cos T$, antiperiodic with period $\Delta T = \pi$ according to the origin $T_0 = \pi/2$ and, therefore, $\mathbf{M}_T\{\phi_1\} = 0$.

The time averages μ_0, μ_2 can be, at least for $m \geqq 4/3$, calculated by numerical quadratures without difficulties. However, it is for practical calculations, *e.g.* for a running treatment of data, rather tedious. The following approximate expression of functions $\mu_0(m, Fr)$, $\mu_2(m, Fr)$ may be recommended for calculation on handheld calculators:

$$\mu_0(m, Fr) \approx \begin{cases} -1 + H(\tfrac{1}{2}(m + 2), 0, \tfrac{1}{2}(m + 1);\ Fr^2);\ Fr \leqq 1 \\ -1 + \alpha_0(m) Fr^{m-1} H(\tfrac{1}{2}(m + 2), \tfrac{1}{2}, \tfrac{1}{2}(m + 1);\ Fr^{-2});\ Fr \geqq 1 \end{cases} \tag{24a,b}$$

$$\mu_2(m, Fr) \approx \begin{cases} \tfrac{1}{4}(m - 1) H(\tfrac{3}{2}m, 1, \tfrac{3}{2}(m - 1);\ Fr^2);\ Fr \leqq 1 \\ \alpha_2(m) Fr^{3m-5} H(\tfrac{3}{2}m, \tfrac{1}{2}, \tfrac{3}{2}(m - 1);\ Fr^{-2});\ Fr \geqq 1, \end{cases} \tag{25a,b}$$

where

$$\alpha_0(x - 1) = \frac{(x - 1)\,\Gamma(\tfrac{1}{2}(x - 1))}{\sqrt{\pi}\,\Gamma(\tfrac{1}{2}x)} \doteq \sqrt{\left[\frac{2x}{\pi e}\left(1 + \frac{1}{x}\right)^x\right]}\left(1 - \frac{1}{6x(1 \cdot 116 + x)}\right), \tag{26}$$

$$\alpha_2(m) = \frac{m-1}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}(3m-2))}{\Gamma(\frac{1}{2}(3m-1))} = \frac{m-1}{2(3m-2)} \alpha_0(3m-2) \tag{27}$$

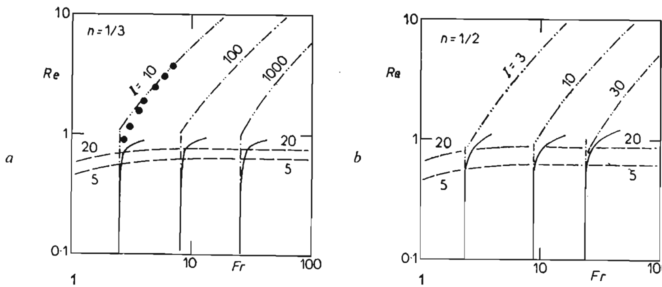
and where H stands for a common polynomial form of the type

$$H(a, b, c; x) = 1 + \sum_{k=1}^{\infty} x^k \prod_{j=1}^{k} \left(\frac{a}{j+b} - 1\right)\left(\frac{c}{j} - 1\right) =$$

$$= 1 + x\left(\frac{a}{1+b} - 1\right)\left(\frac{c}{1} - 1\right)\left\{1 + x\left(\frac{a}{2+b} - 1\right)\left(\frac{c}{2} - 1\right)\{1 + \ldots\}\right\} \tag{28}$$

RESULTS AND DISCUSSION

The main result of the paper is the asymptotic representation of function $I = I(Re, Fr, m)$ for $Re \to 0$ expressed by Eq. (22) and construction of the practically



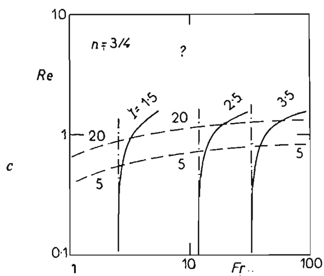


FIG. 1

Topographical projection, I = const., of the function $I = I(Re, Fr, m)$ for $n = 1/3$ (a), $n = 1/2$ (b), $n = 3/4$ (c). Vertical dot-and-dashed lines: asymptotes $Re = 0$; full lines: creeping asymptotes for $Re \ll 1$ according to (22); declined dot-and-dashed lines: boundary-layer asymptotes for $Re \gg 1$, $Fr/Re \gg 1$ according to the foregoing work[5]; horizontal dashed lines: lines of constant relative inertia effect $\psi_2 = 0{\cdot}05$ and $\psi_2 = 0{\cdot}2$. The points in (a) show experimental and numerically simulated data according to the already published work[3] usable algorithmes for calculation of functions $\mu_0(F, m)$, $\mu_2(Fr, m)$. Although it seems at first sight, that μ_0, μ_2 could be expressed by incomplete beta functions, this is not true.

It is obvious from the structure of the asymptotic representation (*22*), that at extremely high $m = 1/n$, the dependence of I on Re according to Eq. (*22*) has a jump character (*e.g.* for $n = 0{\cdot}1$ it holds according to Eq. (*22*) $I \approx 1 + \mu_0 - \mu_0 \varepsilon_2 Re^{20}$). Therefore for the calculation of I the representation (*22*) may be of practical interest only at higher values of flow index, say $n > 0{\cdot}5$, and only in a very narrow range, $Re \ll 1$. Topographic projection of function $I(Re, Fr, m)$ for three chosen values of flow index documents this fact in Fig. 1. The results of the boundary-layer approximation[3-5] are depicted in Fig. 1 in addition to the asymptotic representation (*22*) and to the limiting value of I for $Re = 0$. It is obvious, that the asymptotic representation (*22*) is not sufficient to arch the transition from $Re = 0$ up to the region of boundary-layer regime, not even in the region of higher values of flow index, $n \geqq 0{\cdot}5$. To reach this aim, boundary-layer approximations of higher order shall be rather looked for. Fig. 1 shows, that the relation (*22*) is convenient for the calculation of I only in the region of parameters (Fr, Re, m), where correction on the secondary effect of inertia expressed by the term ψ_2,

$$\psi_2 = \varepsilon_2(Fr, m)\, Re^{2m} \tag{29}$$

does not exceed values 0·05 to 0·20, that is 5 to 20% of the total effect FE. The lines $\psi_2 = 0{\cdot}05$ and $\psi_2 = 0{\cdot}20$ are depicted by horizontal dashed lines in Fig. 1. It is not excluded, that the asymptotic representation (*22*) can be used in a broader extension of variables (Re, Fr) for fluids with flow index close to 1. However, neither experimental material nor sufficiently accurate results of a numerical treatment of the problem are at disposal to confirm this conjecture.

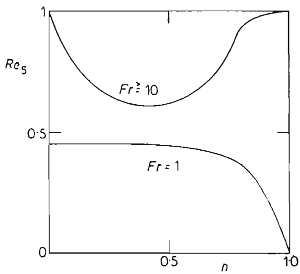


FIG. 2

Dependence of values of Re_5, for which $\psi_2 = 0{\cdot}05$ holds, on flow index for $Fr = 1$ and $Fr = 10$

Therefore for further quantitative study of FE under pulsatile pipe flow the exact demarcation of the region, where the effect of inertia can be certainly neglected, remains, after all, the most interesting result of this paper. If we define this region *e.g.* by demand $\psi_2 = 0{\cdot}05$, then the corresponding critical values of Re, $Re = Re_5$, are shown in Fig. 2 in dependence on Fr for $Fr < 1$. For $Fr < 1$ this critical value is given by the relation

$$Re_5 = \left(\frac{0{\cdot}05m}{4p(m)}\right)^{n/2} Fr^{1-n}\,, \quad Fr < 1 \tag{30a}$$

for $Fr > 10$ this critical value does not already depend on Fr:

$$Re_5 = \left(\frac{0{\cdot}05\alpha_0(m)}{p(m)\,\alpha_2(m)}\right)^{n/2}\,, \quad Fr > 10\,. \tag{30b}$$

It is noteworthy, that the construction of approximative solution in the form of serie (*12*) to the order $k = 2$ inclusive does not have character of the second order differential of the function $W(T, Y; Re, Fr, m)$ with respect to the parameter $M \to 0$, but it is nonuniformly covergent asymptotic expansion about the singular asymptote $M = 0$. Above all to any pair of integers p, q, $p < q$, a line (Y, T) exists, on which the term of the expansion (*12*) of the order p is identically equal zero, and consequently global assumption $M^{q-p}G_p\phi_p/G_q\phi_q = O(M^{q-p})$ is not fulfilled on any surroundings of this line. Secondly, the expansion with respect to the powers M^k is not uniformly convergent to the full extent of the other parameter Fr, as it is already obvious *e.g.*, from the relations (*30a,b*). Thirdly, for $m \to \infty$ the whole approximative scheme breaks down, as the case $m = \infty$, for which an exact solution is known[6], has quite different functional character. Already from those, in passing above indicated reasons it is obvious, why it is not too interesting to make an effort for an assessment of further terms of the expansions (*12*) and (*22*) in the frame of the given approximation scheme.

LIST OF SYMBOLS

$a = a_0 + P_0/(\varrho\omega^2)$	
$g = g_z + P_S/\varrho$	
r, z	radial and cylindrical coordinate
t	time
$T = \omega t$	
$W = (\tau - \varrho g r/2)/(\varrho a \omega^2 r/2)$	
$Y = r/R$	
$\tau = \tau_{rz}$	shear stress

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